

Maxwell equation, Shroedinger equation, Dirac equation, Einstein equation defined on the multifractal sets of the time and the space

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1 abstract

What forms will have an equations of modern physics if the dimensions of our time and space are fractional? The generalized equations enumerated by title are presented by help the generalized fractional derivatives of Riemann-Liouville.

2 Introduction

In the articles [1]-[6] the generalized fractional Riemann-Liouville derivatives (GFD) are determined and the fractal theory of time and space (and some others physical questions) basing on the using GFD for functions defined on a multifractal sets are presented. The multifractal time and space sets are characterized by fractal dimensions $d_t(\mathbf{r}(t), t)$ and $d_{\mathbf{r}}(t(\mathbf{r}), \mathbf{r})$. In this paper the generalization of main equations of the modern physics are presented for the multifractal time and space in the frame of multifractal model of time and space presented in [1]-[6]. These equations gives in a little corrections

for the known equations for the case when the fractal dimensions (FD) of time d_t and space d_r are $d_t = 1 + \varepsilon(\mathbf{r}(t), t)$ (and so on d_r) and the FD are slightly differs from unity, i.e. $|\varepsilon| \ll 1$, that is valid for small densities of Lagrangians in points t, \mathbf{r} , i.e. for weak forces in the domain of space and time near \mathbf{r}, t . All the equations may be received by means of the principle of minimum of fractal dimensions functional (see [1]) and from this principle the generalized Euler's equations may be write down. We use more simple method in this article, consisting in the replacing the ordinary derivatives in the known equations by GFD (it may be ground by comparison with the generalized Euler equations). Before receiving the equations we remind the main definitions and designations of the theory [1]-[4]:

Generalized fractional derivatives (GFD):

We begin from remembering of the fractional Riemann-Liouville derivatives definitions [7]:

$$D_{+,t}^d f(t) = \left(\frac{d}{dt} \right)^n \int_a^t dt' \frac{f(t')}{\Gamma(n-d)(t-t')^{d-n+1}} \quad (1)$$

$$D_{-,t}^d f(t) = (-1)^n \left(\frac{d}{dt} \right)^n \int_t^b dt' \frac{f(t')}{\Gamma(n-d)(t'-t)^{d-n+1}} \quad (2)$$

Let a function $f(t)$ of variable t is defined on multifractal set S_t which consist from multifractal subsets $s_i(t_i)$. We shall see subsets $s_i(t_i)$ as the "points" t_i (with a continuous distribution for different multifractal subsets $s_i(t_i)$ of multifractal set S_t ordered by values of t . Let the function $d(t_i) = d(t)$ is continuous and describes their fractional dimensions (in some cases coinciding with local fractal dimensions of set S_t as function t . For the elementary generalization the definitions (1)-(2) are used physical reasons and variable t is interpreted as a time. For a continuous functions $f(t)$ (the generalized functions defined on the class of the finitary functions (see[8]), the fractional derivatives of the Riemann - Liouville are continuous also. So for infinitesimal intervals of time the functionals (1)-(2) will vary on an infinitesimal quantity. For the continuous function $d(t)$ the changes it thus also will be infinitesimal. It allows, as the elementary generalization (1) that is suitable for describe the changes the function $f(t)$ defined on multifractal subsets $s(t)$ (as well as in the (1)-(2)), to take into account the summary influence of a kernel of integral $(t-t')^{-d(t)-n+1} \Gamma^{-1}(n-d(t))$, depending from

$d(t)$, on the $f(t)$ in all points of integration and, instead of (1)-(2) to write the integral which takes into account all this influences. Thus, we enter the following definitions (generalized fractional derivatives and integrals (GFD)), that account also dependence $d(t)$ from time and vector parameter $\mathbf{r}(t)$ (i.e. $d_t \equiv d_t(\mathbf{r}, t)$):

$$D_{+,t}^{d_t} f(t) = \left(\frac{d}{dt} \right)^n \int_a^t dt' \frac{f(t')}{\Gamma(n - d_t(t'))(t - t')^{d_t(t') - n + 1}} \quad (3)$$

$$D_{-,t}^{d_t} f(t) = (-1)^n \times \left(\frac{d}{dt} \right)^n \int_t^b dt' \frac{f(t')}{\Gamma(n - d_t(t'))(t' - t)^{d_t(t') - n + 1}} \quad (4)$$

In (3)-(4), as well as in (1)-(2), a and b stationary values defined on an infinite axis (from $-\infty$ to ∞), $a < b$, $n - 1 \leq d_t < n$, $n = \{d_t\} + 1$, $\{d_t\}$ - the integer part of $d_t \geq 0$, $n = 0$ for $d_t < 0$. The only difference the (3)-(4) from the (1)-(2) is: $d_t = d_t(\mathbf{r}(t), t)$ - fractional dimensions (further will be used for it terms " fractal dimensions " (FD) or " the global fractal dimension (FD)" of subset s_t) is the function of time and coordinates, instead of stationary values in the (1)-(2). Similar to (1)-(2), it is possible to define the GFD, (that coincides for integer values of fractional dimensions $d_{\mathbf{r}}(\mathbf{r}, t)$ with derivatives respect to vector variable \mathbf{r}) $D_{+,\mathbf{r}}^{d_{\mathbf{r}}} f(\mathbf{r}, t)$ respect to vector $\mathbf{r}(t)$ variables (spatial coordinates). We pay attention, that definitions (1)-(2) are a special case of Hadamard derivatives [9].

2. The connection between the fractional dimensions (FD)of time and space with Lagrangian functions of energy densities read:

$$d_t = 1 + \sum_{i,\alpha} \beta_{i,\alpha} L_{i,\alpha}(t, \mathbf{r}, \Phi_i, \psi_i) \quad (5)$$

In (5) α takes value: $\alpha = t, \mathbf{r}$. More complicated dependencies of d_{α} at $L_{\alpha,i}$ are considered in [1]. Note that relation (5) (and similar expression for $d_{\mathbf{r}}$ does not contain any limitations on the value of $\beta_i L_{i,\alpha}(t, \mathbf{r}, \Phi_i, \psi_i)$ unless such limitations are imposed on the corresponding Lagrangians, and therefore d_t can reach any whatever high or small value.

3. Let's write now the equations enumerated in title in fractal time and space by using GFD:

a) Maxwell equations:

$$\sum_{i=1}^3 D_{-,i,r}^{d_r} D_{+,i,r}^{d_r} A_j(x) - \frac{1}{c^2} D_{-,t}^{d_t} D_{+,t}^{d_t} A_j(x) + m^2 A_j(x) = \frac{4\pi}{c} j_j(x), \quad (6)$$

$$j_j = e D_{+,j,t}^{d_j} r_i$$

$$D_{+,j,r}^{d_j} A_j(x) = 0 \quad (7)$$

In (6)-(7) FD d_j is equal to $d_{\mathbf{r}}$ for $j = 1, 2, 3$ and d_t for $j = 0$ and introduced the mass of foton for providing existence of GFD on infinity (then it must be select equal zero).

b) Shredinger equation

$$-i\hbar D_{+,t}^{d_t} \psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} D_{-,r}^{d_r} D_{+,r}^{d_r} \psi(\mathbf{r}, t) - e^2(\mathbf{r}, t) \psi(\mathbf{r}, t) \quad (8)$$

where in $D_{-,r}^{d_r}, D_{+,r}^{d_r}$ operators ∇ replaced by operators $\nabla \rightarrow \nabla - ie/\hbar c \mathbf{A}$

c) Dirac equation

$$[i\gamma_i(D_{+,i}^{d_i} - ieA_i(x)) - m]\psi(x) = 0 \quad (9)$$

where γ_i are Dirac matrices. It is necessary to make the difference for GFD $D_i^{d_i}$ with respect to t or with respect to \mathbf{r} :

$$D_i^{d_i} = D_t^{d_t}, D_{\mathbf{r}}^{d_r} \quad (10)$$

For atomic electrons the main role plays the electric fields of nucleus. So the density of Lagrangians energy that defined the FG d_t may be selected as

$$d_t = 1 + \beta\Phi(\mathbf{r}, t) \approx 1 + \frac{e^2}{rM_0c^2} \quad (11)$$

where M_0 is the mass of electrical charge body that originate electrical field. It is easy to demonstrate that on the distances of the first Bohr's radius in

atoms the fractional corrections to time dimensions (difference the d_t from unity) have values $\sim 10^{-8}$, so the fractal corrections to electron energy E in atoms will be have values $\sim 10^{-8}E$. It lay out (or in limits domain) of experimental possibilities of the modern experiment.

d) Einstein equation

It is possible to receive the generalization of general relativity equation by using two ways. In the first way it is necessary to introduce a parallel displacement in the Riemann space with fractional dimensions that may be done without difficulties for weak fields (may be it is possible to determine the parallel displacement and for strong fields by the same relations). In that case the carrier of a measure is the Riemann space and we obtain the determination for covariant derivatives in Riemann space with fractional dimensions

$$D_{\pm,\alpha}^{d_i} t^{\mu\nu} = D_{\pm,\alpha}^{d_i} t^{\mu\nu} + \gamma_{\alpha\beta}^{\nu} t^{\mu\beta} \quad i = t, r \quad (12)$$

where $t^{\mu\nu}$ tensor and $\gamma^{\mu\nu}$ metric tensor the Riemann "four-dimension space with fractional dimensions", $D_{\pm,\alpha}^{d_i}$ is GFD, $\gamma_{\alpha\beta}^{\nu}$ are Christoffel's symbols

$$\gamma_{\alpha\beta}^{\nu} = \frac{1}{2} \gamma^{\nu\sigma} (D_{\pm,\alpha}^{d_i} \gamma_{\beta\sigma} + D_{\pm,\beta}^{d_i} \gamma_{\alpha\sigma} + D_{\pm,\sigma}^{d_i} \gamma_{\alpha\beta}) \quad (13)$$

The equations for gravitation field tensor $\tilde{\Phi}^{\mu\nu} = \sqrt{-\gamma} \cdot \Phi^{\mu\nu}$, $\gamma = \det(\gamma_{\mu\nu})$, $\tilde{t}^{\mu\nu} = \sqrt{-\gamma} \cdot t^{\mu\nu}$, L - is a scalar density of matter (see in details (6)-(7)) than read

$$\gamma^{\alpha\beta} D_{-,\alpha}^{\nu,d_i} D_{+,\beta}^{\nu,d_i} \Phi^{\mu\nu} + b^2 \tilde{\Phi}^{\mu\nu} = -\lambda \frac{\delta L}{\delta \gamma^{\mu\nu}} = \lambda \tilde{t}^{\mu\nu} (\gamma^{\mu\nu}, \Phi_A) \quad (14)$$

where b is a constant value that necessary to introduce for using more broad sets of functions with GFD and it after calculations may be put zero. The equation for curvature tensor (with GFD) have an usual form

$$R^{\mu\nu} - \frac{1}{2} \gamma^{\mu\nu} R = \frac{8\pi}{\sqrt{-g}} T^{\mu\nu} \quad (15)$$

$$D_{\pm,\mu}^{d_i} \tilde{g}^{\mu\nu} = 0 \quad (16)$$

The equation (16) describes the boundary conditions for $g^{\mu\nu}$ on the Universe surface. Stress, that equations (12)-(14) describe gravitation fields in the

Riemann space with fractional dimensions, i.e. the carrier of measure is the Riemann space. For the case of weak fields the generalized covariant derivatives may be represented as (see [1])

$$D_{\pm,\alpha}^{d_i} t^{\mu\nu} \approx ' D_{\pm,\alpha}^{d_i} t^{\mu\nu} + '' D_{\pm,\alpha}^{d_i} t^{\mu\nu} \quad (17)$$

The $'D_{\pm,\alpha}^{d_i}$ in (17) describes the contribution from FD of time and space, the member $''D_{\pm,\alpha}^{d_i}$ describes the contribution from Riemann space with integer dimensions.

The second way for describing the gravitation fields in the fractal time and space (by GFD using) consists in the other the measure carrier selection. It is more simplest way to select the measure carrier as the flat four-dimensions pseudo Euclidean Minkowski space. In that case may be used (as a base) the system of reference which coincide for FD equal to unit with Cartesian system of reference (we remember that in the fractal theory of time and space there are an absolute systems of reference). The equations of the gravitation in that case will be analogies the equation of the theory [10] in which all derivatives replaced on GFD and metric tensor $\gamma^{\mu\nu}$ are consist the functions (functionals) originated by fractional dimensions (i.e. it must be the function of L - Lagrangian energy densities of gravitation fields). These equations have the form

$$\gamma^{\alpha\beta} D_{-,\alpha}^{\nu,d_i} D_{+,\beta}^{\nu,d_i} \Phi^{\mu\nu} + b^2 \tilde{\Phi}^{\mu\nu} = -\lambda \frac{\delta L}{\delta \gamma^{\mu\nu}} = \lambda \tilde{t}^{\mu\nu}(\gamma^{\mu\nu}, \Phi_A) \quad (18)$$

The equation (18) differs from equation (14) by three aspects:

- a) the metric tensor $\gamma^{\mu\nu}$ now determined on the Minkowski space with fractional dimensions;
- b) it differs by dependencies of metrics tensor $\gamma^{\mu\nu}$ from L (because there are no dependencies in $\gamma^{\mu\nu}$ of L originating through the Riemann metric tensor), there are only dependencies originating through FD;
- c) the reason of appearance of the dependencies the $\gamma^{\mu\nu}$ at L lay in the originate it by the fractal dimensions of time and space. If FG are integer the (17) coincide with equation of the theory [10]. For weak fields GFD may be represented only by FD covariant derivatives (only one member in right part of (17)) and in that case may be represented by metric tensor $g^{\mu\nu}$ of an "effective" Riemann space with integer dimensions (see [1]). We pay attention that the corresponding results of the theory [10] for connections

between metric tensor $\gamma^{\mu\nu}$ of Minkowski space with "effective" metric tensor $g^{\mu\nu}$ of Riemann space and gravitation tensor are the special case of our theory. In general case the metric tensor of Minkowski space are complicated function of gravitation field tensor.

We leave for readers an interesting task to generalize the equations of quantum gravitation for multifractal time and space.

3 Relations between GFD and ordinary derivatives for d_α near integer values

If $d_\alpha \rightarrow n$ where n is an integer number, for example $d_\alpha = 1 + \varepsilon(\mathbf{r}(t), t)$, $\alpha = \mathbf{r}, t$, in that case it is possible represent GFD by approximate relations (see [1])

$$D_{+,x_\alpha}^{1+\epsilon} f(\mathbf{r}(t), t) = \frac{\partial}{\partial x_\alpha} f(\mathbf{r}(t), t) + \frac{\partial}{\partial x_\alpha} [\varepsilon(\mathbf{r}(t), t) f(\mathbf{r}(t)), t] \quad (19)$$

The replacement in generalized Maxwell equations (6)-(7), Shroedinger equation (8), Dirac equation (9), Einstein equation (13) (and so on) the GFD defined by (3) by approximation of GFD defined in the (19) gives a possibility to solve numerous tasks in fractal space and time and calculate the corrections from fractional dimensions for these tasks.

4 Conclusion

1. In this paper were presented the main equation of modern physics defined on multifractal sets of time and space. In case integer dimensions of time and space all of them coincide with the known equation. The value of correction to integer dimensions of time and space in conditions of Earth are very small. So, the correction to dimension of time that gives the gravitational field of Earth on the ground of Earth is equal $\sim 10^{-9}$. The corrections from electric field nucleus on atomic distances from nucleus are $\sim 10^{-8}$. So, it may be neglected by these corrections, but only in the cases of weak fields. In the case of strong fields all generalized equations becomes in the integral fractional equations. The last don't have singularities and only this fact originate the interest to these equations. We pay attention that in the fractal theory of time and space our Universe is the open system (the statistical theory of open system

are in [11],[12]

2.Can these equation be used for strong fields (i.e. for FD that differs a lot of from integer values). This question now is open. We may only say that answer on this question concerns with answer on other question: is it possible to use the method of ordinary Lagrangians of quantum and classical theories for describing the strong fields? If answer on last question is positive, there are a hope that the positive answer for the first question will be correct.

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